EXPANSION OF A PISTON IN WATER

(O RASSHIRENII PORSHNIA V VODE)

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1. The problem of piston expansion with constant velocity in an ideal gas has been solved by Sedov [1, 2] and also by Taylor [3].

We assume that the expansion at constant velocity of a spherical, cylindrical or plane piston takes place in some ideal fluid. Sedov [2] has demonstrated that the problem of transient motion in the medium between the piston and the shock wave originating herefrom remains a self-similar one whatever the internal energy of the medium. The latter can be expressed in the following general form:

$$\varepsilon(p,\ \rho) = \frac{p}{\rho_1} f\left(\frac{\rho}{\rho_1},\ \frac{p}{p_1}\right) + \text{const}$$

In this expression p_1 and ρ_1 are the parameters which determine the problem. When the piston expands at constant velocity, only the two parameters, p_1 and ρ_1, of all the parameters of the problem, possess independent dimensions; the dimension of piston velocity is

$$[U] = \left[\sqrt{\frac{p_1}{\rho_1}}\right]$$

Sedov has also shown [2] that if the internal energy of the medium $\epsilon(p, \rho)$ is expressible as

$$\varepsilon(p, \rho) = \frac{p}{\rho_1} \varphi\left(\frac{\rho}{\rho_1}\right) + \text{const} \tag{1.1}$$

where ϕ is an arbitrary function of its argument, then the following relations are valid.

1. The equation for the adiabatic curve is in this form;

$$p = \Psi(S)\,\chi\left(\frac{\rho}{\rho_1}\right) \tag{1.2}$$

where ψ is some function of the entropy S which can be found from supplementary physical considerations, whilst the connection between functions

$\phi(R)$ and $\psi(R)$ comes out of the formulas

$$\varphi(R) = \frac{1}{\chi(R)}\left(C + \int \frac{\chi(R)\,dR}{R^2}\right), \qquad \chi(R) = \frac{C}{\varphi(R)} \exp \int \frac{dR}{R^2 \varphi(R)} \tag{1.3}$$

where C is an arbitrary constant.

2. The equation of state, involving condition (1.1), should be

$$T = \exp \int \frac{dR}{R^2 \varphi(R)}\, \Phi \left[p\varphi(R) \exp\left(-\int \frac{dR}{R^2 \varphi(R)}\right)\right] \qquad \left(R = \frac{\rho}{\rho_1}\right) \tag{1.4}$$

where Φ is some function related to Ψ by the equation:

$$\Psi'(S) = \rho_1 \Phi[\Psi(S)] \tag{1.5}$$

The problem of a strong point explosion, under condition (1.1), has been discussed in [4].

We will now study the problem of the piston expanding with constant velocity in an ideal medium with the assumption that the internal energy of the medium can be defined by formula (1.1).

Bearing in mind the expression (1.2), the one-dimensional equations of transient motion in an ideal compressible fluid become

$$\begin{gathered}\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0 \\ \frac{\partial}{\partial t}\frac{p}{\chi(\rho/\rho_1)} + v\frac{\partial}{\partial r}\frac{p}{\chi(\rho/\rho_1)} = 0\end{gathered} \tag{1.6}$$

The condition at the piston surface r is:

$$r = \bar{r}_* = Ut \quad \text{or} \quad v = U \tag{1.7}$$

where U is the velocity of the piston.

Conditions on the shock wave which propagates through the undisturbed medium, incorporating assumption (1.1), can be written down thus

$$-\rho_1 D = \rho_2(v_2 - D), \qquad \rho_1 D^2 + p_1 = \rho_2(v_2 - D)^2 + p_2 \tag{1.8}$$

$$\frac{1}{2}D^2 + \frac{p_1}{\rho_1}\varphi(1) + \frac{p_1}{\rho_1} = \frac{1}{2}(v_2 - D)^2 + \frac{p_2}{\rho_1}\varphi\left(\frac{\rho_2}{\rho_1}\right) + \frac{p_2}{\rho_2}$$

Suffix 1 denotes conditions in front of the shock wave, suffix 2 those behind it. D is the velocity of the shock wave.

From considerations of the similarity of the problem the required characteristics of the motion can be sought in this form

$$v = \frac{r}{t} V(\lambda), \qquad \rho = \rho_1 R(\lambda), \qquad p = \rho_1 \frac{r^2}{t^2} P(\lambda) \tag{1.9}$$

where

$$\lambda = \frac{r}{r_2}, \qquad r_2 = \frac{U}{\lambda_*} t, \qquad \lambda_* = \frac{r_*}{r_2} \tag{1.10}$$

In the above, r_2 is the radius of the shock wave and λ_* is the dimensionless piston radius.

Using formulas (1.9) and (1.10) we make equations (1.6) non-dimensional;

$$(1 - V)\frac{dV}{d\ln\lambda} - \frac{1}{R}\frac{dP}{d\ln\lambda} = V^2 - V + 2\frac{P}{R} \tag{1.11}$$

$$-\frac{dV}{d\ln\lambda} + (1 - V)\frac{d\ln R}{d\ln\lambda} = \nu V \tag{1.12}$$

$$\frac{d}{d\ln\lambda}\ln\frac{P}{\chi(R)} = -2 \tag{1.13}$$

Integrating (1.13) and making use of the second expression in (1.3) we arrive at the integral expressing degree of "adiabaticity".

$$P = \frac{C}{\lambda^2\varphi(R)}\exp\int\frac{dR}{R^2\varphi(R)} \tag{1.14}$$

Using expression (1.14), after some rearrangement we bring equations (1.11) and (1.12) into the following form;

$$\left\{(1 - V)^2 - \left[P + \frac{1-\nu}{2}RV(V-1)\right]\frac{1}{\varphi(R)}\left[\frac{1}{R^2} - \varphi'(R)\right]\right\}\frac{d\ln R}{dV} =$$
$$= -\frac{1}{2}(1-\nu)V(V-1)\frac{d\ln P}{dV} \tag{1.15}$$

$$(1 - V)\frac{d\ln R}{dV} = \nu V\frac{d\ln\lambda}{dV} + 1 \tag{1.16}$$

By reason of (1.2) and the second of the expressions (1.3) we arrive at the following expression for the square of the velocity of sound

$$a^2 = \frac{r^2}{t^2}\frac{P}{\varphi(R)}\left[\frac{1}{R^2} - \varphi'(R)\right] \tag{1.17}$$

Introducing the dimensionless function

$$w = \frac{P}{\varphi(R)}\left[\frac{1}{R^2} - \varphi'(R)\right] \tag{1.18}$$

we transform the system of equations (1.15), (1.16) and (1.14) into the following

$$\frac{d\ln w}{dV}=\frac{1}{[\nu w-(1-V)^2]}\left\{\frac{2}{V}[w-(1-V)^2]+\right.$$
$$\left.+(\nu-1)(1-V)R\left[\frac{1}{\varphi(R)}\left(\frac{1}{R^2}-\varphi'(R)\right)+\frac{d}{dR}\ln\frac{1}{\varphi(R)}\left(\frac{1}{R^2}-\varphi'(R)\right)\right]\right\} \tag{1.19}$$

$$\frac{d\ln R}{dV}=\frac{(\nu-1)(1-V)}{\nu w-(1-V)^2} \tag{1.20}$$

$$\lambda=\frac{1}{\varphi(R)}\left(\frac{C}{w}\exp\int\frac{dR}{R^2\varphi(R)}\left[\frac{1}{R^2}-\varphi'(R)\right]\right)^{1/2} \tag{1.21}$$

Converting the shock conditions (1.8) to nondimensionals with the help of formulas (1.9) and (1.10), and making use of (1.18), we find the quantities R_2, w_2, w_1 in the function of V_2

$$w_2=\frac{-V_2[(1-V_2)^2-\varphi'(R_2)][V_2+2\varphi(1)]}{2\varphi(R_2)\{\varphi(R_2)-V_2-\varphi(1)\}},\qquad R_2=\frac{1}{1-V_2} \tag{1.22}$$
$$w_1=\frac{[1-\varphi'(1)]}{2\varphi(1)}\,\frac{V_2\{V_2-2\varphi(R_2)\}}{\{\varphi(R_2)-V_2-\varphi(1)\}},$$

In view of (1.7) and (1.9), the condition at the piston in dimensionless form can be written

$$V=1 \tag{1.23}$$

Thus, for the case when the internal energy of the fluid is in the form (1.1), the problem reduces to integrating the system of equations (1.19) and (1.20) so as to find the two functions w and R, which satisfy the boundary conditions (1.22) and (1.23).

Having determined w and R, we can find λ in its final form from equation (1.21). The arbitrary constant C in (1.21) is so determined that when $R = R_2$, λ is unity.

Now let us deal with the case $\nu \neq 1$, Equations (1.20) and (1.19) show that for equation (1.19) to be studied qualitatively and integrated independently of (1.20), it is sufficient for the following equation to be valid:

$$\frac{1}{\varphi(R)}\left[\frac{1}{R^2}-\varphi'(R)\right]+\frac{d}{dR}\left(\ln\frac{1}{\varphi(R)}\left[\frac{1}{R^2}-\varphi'(R)\right]\right)=\frac{\varkappa-1}{R} \tag{1.24}$$

where χ is an arbitrary constant. In this case R does not enter the R.H.S. of equation (1.19) and is determined from (1.20) by quadrature. On integrating equation (1.24) we find

$$\varphi(R) = \frac{R^{\varkappa} + kR + b(\varkappa - 1)}{(\varkappa - 1) R (R^{\varkappa} - b)} \tag{1.25}$$

Here k and b are arbitrary constants.

Because of the second of expressions (1.3), the equation of the adiabatic curve becomes

$$p = \Psi(S)(R^{\varkappa} - b) \tag{1.26}$$

The temperature T can be found from equation (1.4)

2. It is well known from experience that at very high pressures (of the order of ten to one hundred thousand atmospheres) water, like other liquids, is no longer incompressible [5,6].

There is no generally accepted equation of state for water. Some authors [7,8] claim that the adiabatic water equation is:

$$p = \Psi(S)(\rho^{\varkappa} - \rho_0^{\varkappa}) \tag{2.1}$$

in which the value of constant κ is close to 7.

Experiments on the density and temperature of water at high pressures [5,6] reveal that at constant density the pressure is a linear function of the temperature. Thus in equation (1.4) we get Φ as a linear function of its argument. Using (1.4), (1.5) and (1.3) we find the adiabatic equation, the equation of state and an expression for the internal energy of water

$$p = [B + e^{A\rho_1(S-S_0)}](R^{\varkappa} - b) \qquad \left(b = \left(\frac{\rho_0}{\rho_1}\right)^{\varkappa}\right) \tag{2.2}$$

$$T = \frac{A\,[R^{\varkappa} + kR + b(\varkappa - 1)]}{(\varkappa - 1) R}\left[\frac{p}{R^{\varkappa} - b} - B\right] \tag{2.3}$$

$$\varphi(R) = \frac{R^{\varkappa} + kR + b(\varkappa - 1)}{(\varkappa - 1) R (R^{\varkappa} - b)} \tag{2.4}$$

In these expressions $\kappa > 1$, b, A, B, k are constants determined from experiment.

On comparing (2.2) with (1.26) and (2.4) with (1.25) we see that under conditions (1.24), equations (1.19)-(1.21) and boundary conditions (1.22) and (1.23) describe the motion of water displaced by a piston expanding with constant velocity. Using condition (1.24), we transform equations (1.19)-(1.21) into

$$\frac{dw}{dV}=\frac{2w}{V}\frac{[w-(V-1)(lV-1)]}{[\nu w-(1-V)^2]} \tag{2.5}$$

$$\frac{d\ln\lambda}{dV}=\frac{-[w-(1-V)^2]}{V[\nu w-(1-V)^2]} \tag{2.6}$$

$$R=C_1(w\lambda^2)^{1/(\varkappa-1)} \tag{2.7}$$

where C_1 is an arbitrary constant; l denotes

$$l=1+\frac{1}{2}(\nu-1)(\varkappa-1) \tag{2.8}$$

From (1.18) and (2.4) we find

$$w=\frac{\varkappa PR^{\varkappa-1}}{R^{\varkappa}-b} \tag{2.9}$$

If the constants b and k in equations (1.25) and (1.26) are zero, we are reduced to the problem of an expanding piston in gas with adiabatic index κ. The whole field of integral curves of equation (2.5) has been studied in detail and the corresponding problems have been completely solved by Sedov [1,2]. As a result of the foregoing, however, it appears that equations (2.5)-(2.7) have the same form for water as for gas (only with the difference that for water w is given by (2.9) whilst with gas $b = 0$, i.e. $w = \kappa P/R$ in this formula). It therefore follows that the field of integral curves of equation (2.5) will have the same appearance as in the case of gas [1]. Singular points of this equation are:

nodes: O (0,0), A (0,1), C (1,0)

saddle points: B (V^0, w^0), D ($\pm\infty$, ∞)

and

$$V^0=\frac{2}{2+\nu(\varkappa-1)}$$

$$w^0=\frac{\nu(\varkappa-1)^2}{[2+\nu(\varkappa-1)]^2}$$

The parabola $w = (V-1)^2$ corresponds to a weak explosion; thus it is not possible to proceed from the point where $V = 1$, corresponding to the piston, to the point O (0,0), which corresponds to the point at infinity in space, in a continuous manner along an integral curve; it is only possible to get there by means of a jump.

We can get the solution thus: from point $V = 1$, $w = w^*$ (where w^* is some given constant) we move along the integral curve to point (V_2, w_2) from which we reach point (O, w_1) of axis w, by a jump corresponding to the outer edge of the shock. We then move along the integral straight line $V = 0$ to point O. Thus for each given value of w_1 we get its associated integral curve.

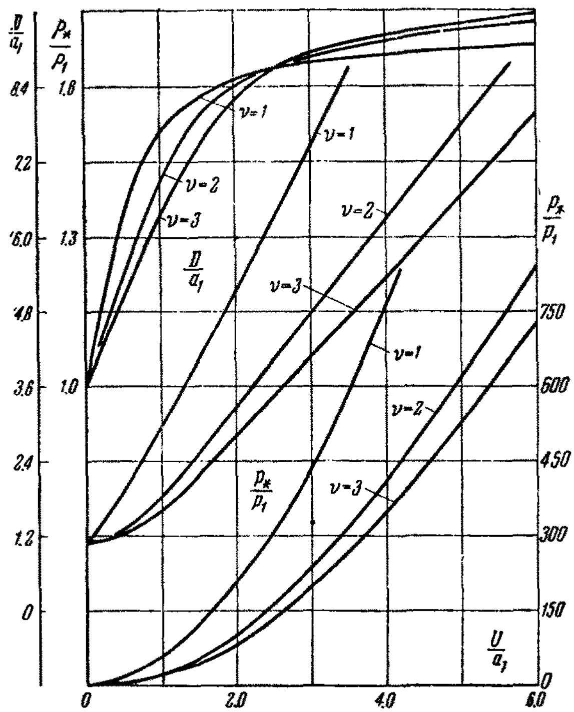

Fig. 1.

In view of the fact that points (1, w^*) corresponding to the piston are ordinary points, parameter λ acquires at this point some constant value less than unity (assuming that $\lambda = 1$ on the shock wave).

Because of (1.18) and (1.25), the conditions (1.22) yield the following expressions of R_2, w_2 and w_1 in terms of V_2

$$R_2 = \frac{1}{1 - V_2} \tag{2.10}$$

$$w_2 = -\frac{\varkappa(\varkappa - 1)}{2}\,\frac{V_2(1 - V_2)[V_2 + 2\varphi(1)]}{\{[1 - (\varkappa - 1)\varphi(1)][1 - (1 - V_2)^{\varkappa}] - \varkappa V_2\}} \tag{2.11}$$

$$w_1 = \frac{\varkappa}{2(1 - b)}\,\frac{\{-2 + (\varkappa + 1)V_2 + B(1 - V_2)^{\varkappa} - (\varkappa - 1)b(1 - V_2)^{\varkappa + 1}\}}{\{[1 - (\varkappa - 1)\varphi(1)][1 - (1 - V_2)^{\varkappa}] - \varkappa V_2\}} \tag{2.12}$$

where

$$B = 2 - (\varkappa - 1)[2(1 - b)\varphi(1) - b] \tag{2.13}$$

Assuming that $\lambda = 1$ on the shock wave, we find from (2.7)

$$C_1 = R_2 w_2^{-\frac{1}{\varkappa - 1}}$$

Moreover, P is linked with w and R in conformity with

$$P = \frac{1}{\varkappa} \frac{w(R^{\varkappa} - b)}{R^{\varkappa - 1}} \tag{2.14}$$

We thus have the system of four equations (2.5), (2.6), (2.7) and (2.14) for determining R, P, V and w as functions of λ.

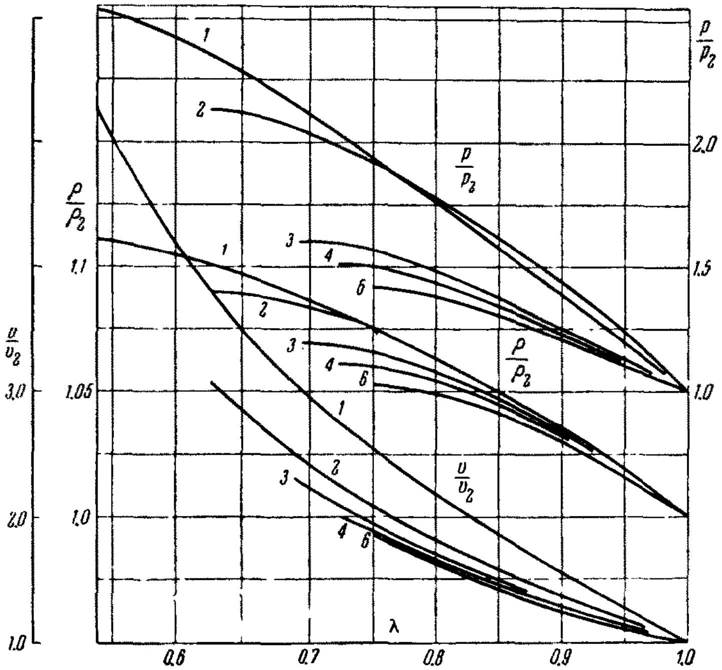

Fig. 2.

The chief difficulty in solving this system lies in the numerical integration of equation (2.5) with boundary conditions (2.11) and (1.23). Log λ can be found by quadrature from equation (2.6) after determining $w(V)$. R and P can be found from equations (2.7) and (2.14) respectively.

Equations (2.5)-(2.8), (2.14), (1.9) and (1.10) reveal that with a plane piston, moving inside a cylindrical tube full of water, a region of constant velocity, constant density and constant pressure is created between the piston and the shock wave.

Fig. 1 shows graphs of functions p_*/p_1, ρ_*/ρ_1 and D/a_1, dimensionless pressure and density on the piston and the dimensionless shock wave velocity, respectively as a function of the dimensionless piston velocity U/a_1. Figs. 2 and 3 give graphs of v/v_2, ρ/ρ_2 and p/p_2 as functions of λ for various values of parameter w_1. Fig. 2 relates to the case of spherical symmetry, Fig. 3 to cylindrical symmetry.

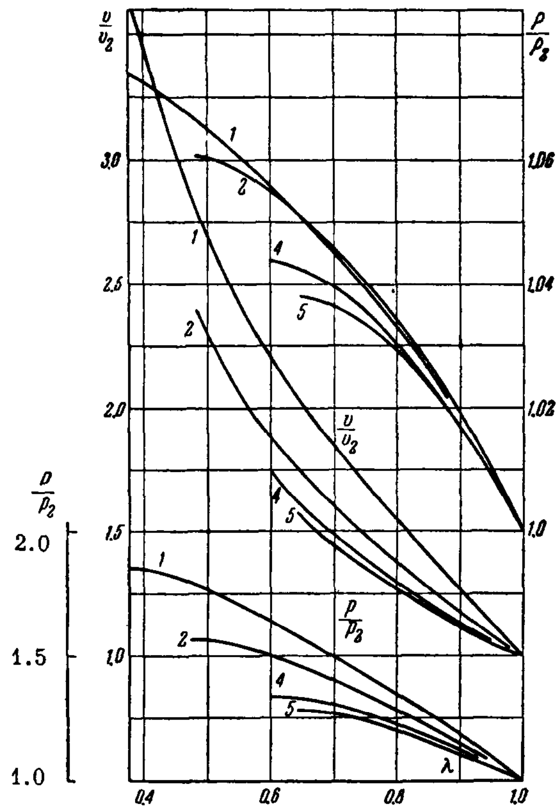

Fig. 3.

The curves are as follows:

curve	1	is	for w_1	=	0.66538
"	2		"		0.37742
"	3		"		0.16720
"	4		"		0.09126
"	5		"		0.01362
"	6		"		0

The constants entering equations (2.2)-(2.4) were obtained from experimental results.

3. Sedov [1,2] gave the solution to the problem of the implosion of ideal gas at a point, and its dispersion outwards. These solutions are given by the integral curves (2.5), where $w = \gamma P/R$. It is clear from the foregoing that the same integral curves can be used for water.

At the initial instant the velocity, density, and pressure, of all the particles of water are equal.

If the initial velocity is directed towards the centre, i.e. it is negative (focussed), water, moving from infinity towards the centre undergoes, first of all, an adiabatic compression, then through the shock, it changes to a state of rest.

If the initial particle velocity is directed outwards (dispersion), for small values of the ratio of initial water velocity to initial velocity of sound $v_1' = v_1/a_1$, there is a nucleus of water which is coming to rest and is expanding in time, and this is separated by the weak shock wave from water moving outwards to infinity.

For some particular value of the quantity $v_1' = v_1'^*$ there is continuous motion, reaching the centre of symmetry. When $v_1' \geqslant v_1'^*$ a hollow region expands at constant velocity, at the boundary of which the density is zero.

The water case differs from that of gas in that for $v_1' > v_1'^*$ there is negative pressure at the centre of symmetry or at the boundary of the hollow space. The surface at which the pressure becomes zero propagates with the particles and cannot, therefore, be regarded as a boundary of the hollow space.

Note that for water, as well as for gas, there is an exact solution to equations (2.5)-(2.7) (corresponding to singular point B)

$$v = \frac{2}{2+\nu(\varkappa-1)}\,\frac{r}{t}$$

$$\rho = \rho_1 C_1 w_*^{\chi}\left(\frac{r}{a_1 t}\right)^{2\chi} \qquad \left(\chi = \frac{1}{\varkappa-1}\right)$$

$$p = \frac{p_1 C_1}{(1-b)}\left[w_*^{\vartheta}\left(\frac{r}{a_1 t}\right)^{2\vartheta} - \frac{b}{C_1^{\varkappa}}\right] \qquad \left(\vartheta = \frac{\varkappa}{\varkappa-1}\right)$$

where a_1 is the sonic velocity in the undisturbed fluid.

$$a_1^2 = \frac{\varkappa p_1}{\rho_1(1-b)}, \qquad w_* = \frac{\nu(\varkappa-1)^2}{[2+\nu(\varkappa-1)]^2}, \qquad \lambda = \frac{r}{a_1 t}$$

Here C_1 is the constant entering equation (2.7).

At the centre of symmetry the pressure is negative, the density and the velocity become zero, whilst at infinite distance these quantities become infinitely great.

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